I. Пресметнете и представете резултата в алгебричен вид:

1. $\left(\frac{\sqrt{3}-i}{1+i}\right)^{24}$

2. $\operatorname{Arctg}\frac{i}{3}$

II. Да се намери аналитична функция $f(z)=u+i.v$, за която е дадено:

1. $v=\frac{y}{x^2+y^2}$, $f(1)=2$

2. $u=2y^2-2x^2-4xy+3x-y$, $f(0)=i$

III. Пресметнете интеграла:

1. $\oint\limits_{c}\frac{\sin\frac{\pi}{4}z}{z^2-1}\,dz$, $c: x^2+y^2=2x+8$

2. $\oint\limits_{c}\sin\frac{1}{z}\,dz$, $c: |z|=1$

II. зад.2 $\quad u = 2y^2 - 2x^2 - 4xy + 3x - y \qquad f(0) = i$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial u}{\partial x} = -4x - 4y + 3 \qquad \frac{\partial u}{\partial y} = 4y - 4x - 1$$

$$\frac{\partial v}{\partial y} = -4x - 4y + 3$$

$$\int \frac{\partial v}{\partial y} = \int -4x - 4y + 3\, dy = \int -4x\, dy - \int 4y\, dy + \int 3\, dy = -4x \int dy - 4 \int y\, dy + 3y = -4xy - \cancel{4}^{2} \cdot \frac{y^2}{\cancel{2}_1} + 3y + \varphi(x)$$

$$v = -4xy - 2y^2 + 3y + \varphi(x) \Big|'_x$$

$$\frac{\partial v}{\partial x} = -4y + \varphi'(x)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$4y - 4x - 1 = -(-4y + \varphi'(x))$$

$$4y - 4x - 1 = 4y - \varphi'(x)$$

$$\varphi'(x) = \cancel{4y} - \cancel{4y} + 4x + 1$$

$$\varphi'(x) = 4x + 1$$

$$\int \varphi'(x) = \int 4x + 1\, dx = \int 4x\, dx + \int 1\, dx = \cancel{4}^{2} \cdot \frac{x^2}{\cancel{2}_1} + x = 2x^2 + x + c$$

$$\varphi(x) = 2x^2 + x + c$$

$$v = -4x - 2y^2 + 3y + 2x^2 + x + c = -3x - 2y^2 + 3y + 2x^2 + c$$

$$f(z) = u + i.v$$

$$f(z) = 2y^2 - 2x^2 - 4xy + 3x - y + i.(-3x - 2y^2 + 3y + 2x^2 + c)$$

III. зад. 1 $\oint\limits_C \frac{\sin\frac{\pi}{4}z}{z^2-1}\,dz$, $C: x^2+y^2=2x+8$

$z^2-1=0$

$(z-1)(z+1)=0$

$z-1=0 \qquad z+1=0$

$z_1=1 \qquad z_2=-1$

z_1 и z_2 са прости полюси

$$Res_1 f(z)=\lim_{z\to1}(z-1)\frac{\sin\frac{\pi}{4}z}{z^2-1}=\lim_{z\to1}\cancel{(z-1)}\frac{\sin\frac{\pi}{4}z}{\cancel{(z-1)}(z+1)}=\lim_{z\to1}\frac{\sin\frac{\pi}{4}z}{z+1}=$$

$$=\frac{\sin\frac{\pi}{4}}{2}=\frac{1}{2}\sin\frac{\pi}{4}$$

$$Res_{-1} f(z)=\lim_{z\to-1}(z+1)\frac{\sin\frac{\pi}{4}z}{z^2-1}=\lim_{z\to-1}\cancel{(z+1)}\frac{\sin\frac{\pi}{4}z}{(z-1)\cancel{(z+1)}}=\lim_{z\to-1}\frac{\sin\frac{\pi}{4}z}{z-1}=$$

$$=\frac{\sin\frac{\pi}{4}}{-2}=-\frac{\sin\frac{\pi}{4}}{2}=-\frac{1}{2}\sin\frac{\pi}{4}$$

$$\oint\frac{\sin\frac{\pi}{4}z}{z^2-1}=2\pi i\,(Res_1 f(z)+Res_{-1}f(z))=2\pi i\left(\cancel{\frac{1}{2}\sin\frac{\pi}{4}}-\cancel{\frac{1}{2}\sin\frac{\pi}{4}}\right)=2\pi i$$