Bulgaria National Olympiad 2025

The exam consists of 6 problems, with 3 problems given each day.

1. Find all real sequences $a_1, a_2, \ldots$ such that
$a_{m^2 + m + n} = a_m^2 + a_m + a_n$
for all positive integers m and n.

2. Exactly n cells of an $n \times n$ square grid are colored black and the remaining cells are white.
The cost of such a coloring is the minimum number of white cells that must be recolored black so that
from any black cell c_0 one can reach any other black cell c_k through a sequence $c_0, c_1, \ldots, c_k$ of black
cells, in which each consecutive pair c_i, c_{i+1} shares an edge. Let $f(n)$ denote the maximum cost over all
initial colorings with exactly n black cells. Find a constant α such that
$(1/3)\, n^\alpha \le f(n) \le 3\, n^\alpha$
for all $n \ge 100$.

3. Let $P(x)$ be a nonconstant monic polynomial with integer coefficients, and let $a_1, a_2, \ldots$ be an infinite
sequence of positive integers. Prove that there exist infinitely many primes p for which there exists a
positive integer n such that p divides $P(n)^{a_n} + 1$.

4. Let ABC be an acute triangle with $AB < AC$, let M be the midpoint of BC, AD the altitude ($D \in BC$),
and H the orthocenter. A circle passes through B and D, is tangent to AB, and meets the circumcircle
of triangle ABC again at Q. The circumcircle of triangle QDH meets line BC again at P. Prove that
lines MH and AP are perpendicular.

5. Let n be a positive integer. Prove that there exists a positive integer a such that exactly $\lfloor n/4 \rfloor$
numbers among $a+1, a+2, \ldots, a+n$ have no square divisor greater than 1.

6. Let $X_0, X_1, \ldots, X_{n-1}$ be $n \ge 2$ given points in the plane, and let $r > 0$ be a real number. Alice and Bob
play the following game. First, Alice constructs a connected graph with vertices at $X_0, X_1, \ldots, X_{n-1}$,
i.e., she connects some points with edges so that from any point one can reach any other by moving
along edges. Then Alice assigns to each vertex X_i a nonnegative real number r_i such that $\sum r_i = 1$.
Then Bob chooses a sequence of distinct vertices $X_{i_0} = X_0, X_{i_1}, \ldots, X_{i_k}$ such that X_{i_j} and
$X_{i_{j+1}}$ are connected by an edge for all $j = 0, \ldots, k-1$. (Note that $k \ge 0$ is not fixed, and the first chosen
vertex must always be X_0.) Bob wins if
$\frac{1}{k+1} \sum_{j=0}^{k} r_{i_j} \ge r$;
otherwise Alice wins. Depending on n, determine the largest possible value of r for which Bob has a
winning strategy.